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The Next Phase of Technology in Math Education: What Happens When Math Software is Truly Easy to Use?

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This article examines the rise, decline, and resurgence of the use of computer algebra systems (CAS) in the mathematics classroom. From the excitement surrounding the technology's potential to revolutionize the teaching of mathematics, it explores the difficulties that prevented this vision from being fully realized, and then introduces a fundamental advancement in the CAS world that finally puts the dream of modern, effective mathematics education within our grasp.

The ambition to make a computer-algebra system the "tool of first recourse for teaching, learning, and doing" mathematics was expressed in the 1987 National Science Foundation (NSF) grant that funded the first step in bringing computers into the classroom at the Rose-Hulman Institute of Technology (RHIT). The first CAS classes there were taught in the fall of 1988, and by 1991 all of the required five math courses for engineering and science students were so taught. By 1995, the Institute adopted a laptop policy, and the dream of a ubiquitous mathematical servant was in the offing.

The calculus-reform movement of the mid-to-late 1980s had induced many pioneers to establish computer labs running some form of CAS, and to try different paradigms for incorporating this technology into the curriculum. At the same time several researchers in math education studied and enunciated a paradigm for using this technology. Captured in the phrase "resequencing skills and concepts," this paradigm suggested that the CAS be used to present concepts "first" before any emphasis on concomitant manipulative skills. Once the concept had been digested, the requisite skills could be explored in the CAS, then mastered to whatever extent necessary for a by-hand environment.

What a glorious vision! Instructors actually teaching mathematics, emphasizing a conceptual understanding of the material, engaging students with ideas, not drills. Students actually learning and adsorbing mathematical ideas, not struggling with manipulations for which they saw no purpose. But what went wrong? Why did this vision fade? As of the mid 1990s, nearly all of the available CAS tools were command-driven, the exception being Derive, discontinued in 2007. This required that students first learn the tool before any of its benefits could be experienced. Initially, the pioneers who first brought CAS into the classroom were willing to "pay the price" of convincing students that learning the language of a CAS was worth the effort. But this approach did not take hold with those who were called on to sustain the vision of the pioneers. A gradual disenchantment with the potential of CAS as the working tool for math instruction set in.

Another obstacle to a wholesale adoption of CAS as a working tool was the restriction the lab environment imposed on the process. If students experience CAS as a lab activity, it fails to be integrated into the curriculum, and cannot become the tool of first recourse. Mandatory lab exercises are seen as extra work, over and above the standard by-hand activities endemic to the traditional courses. Students are wise enough to see this, and they rebel against a technology that seems to increase their workload rather than diminish it. What's on the test is the typical student's concern, and if the test is done with pencil and paper, then those are the skills the student wants to learn.

Three things are necessary for a new technology to take hold: applicability, accessibility and ease-of-use. The new technology must be an improvement over existing modes. It must be readily available. And it has to be easy to implement.

Surely a computing device that can manipulate symbols and perform nearly all the manipulations of the first few years of college mathematics must be seen as a useful tool. That it is useful for doing mathematics is evidenced by its use in industry and commerce today. That it is useful for teaching and learning mathematics is evidenced by its appearance in numerous college math, science and engineering courses.

The laptop computer (and even the advanced "graphing" calculator) promised accessibility; ease-of-use was another matter. At RHIT in the early 90's, where Maple had been adopted as the standard CAS, a 14-page syntax guide was a necessary hand-out at the beginning of the calculus sequence, which itself had to be modified to allow for students to acquire enough mastery of Maple for the actual course material to be presented in that context. The first three weeks of the first term were devoted to those review topics that are embedded in the course. Topics such as solving equations, graphing, finding inverse functions, etc., were used as the training grounds for mastering Maple. (It had been discovered right at the beginning that trying to teach syntax and new concepts at the same time just didn't work.) By the end of the second term, all the traditional topics had been covered, the efficiencies of having a CAS available essentially compensated for spending capital at the beginning of the first term learning Maple. (This changed over time as new releases of Maple became easier and easier to use.)

Note, however, that not all of the effort of learning to use a CAS as a working tool goes into mastering the syntax of its commands. What was observed at RHIT was the student jousting with the structure imposed by the rigidities of a computer language. For example, students wanted to "solve" derivatives, integrals, limits, rather than evaluate them. Well, the CAS has specific commands for these actions, and misuse of a command because of linguistic sloppiness has a way of bringing a student up short. In an environment where the CAS-based course could be avoided, many students did just that. In an environment where the CAS is seen as an extra burden, it does not take hold and thrive.

But some technologies did take hold in the past. High-school math in the 1950s included learning how to use logarithms as a tool for multiplying and dividing cumbersome numbers. In the early 1960s, it was impossible to get through chemistry and physics courses (with their concomitant labs) without knowing how to use a slide rule. By the early 1970s the hand-held calculator was starting to appear as a working tool. In fact, by the mid 1970s, calculators that could, with one entry of a set of data points, compute all the sums, and sums of squares, and sums of products needed for a least-squares fit of a line to the data revolutionized the statistics lab. And anyone who can still take a square-root by hand is an exception, even amongst a gathering of mathematicians.

Logarithms are no longer proposed as a viable working tool for multiplying and dividing numbers. This topic succumbed to the hand-held calculator. Slide rules disappeared from the college classroom with the advent of the personal computer. Least-squares computations are trivialized by modern software. And nobody ever takes a square root by hand. The point is that technologies that save work, are easy to use, and are readily available will displace older and less productive ones. Just think of modern industry. For example, in a furniture factory, boards are planed smooth in a power planer, not with a hand-held jointing plane. The apprenticeship in which the journeyman carpenter mastered the skills needed to true a board was long and exacting. It makes little sense to perpetuate such a parallel process in the academic "factory." Technologies like the CAS that can replace error-prone tedium with instantaneous results must be allowed to displace slower and more labor-intensive approaches.

Students should not be subjected to every step through which their instructors trod on the way to their level of expertise. In other words it is not necessary academically that ontogeny recapitulate phylogeny, a biological theory holding that in developing from embryo to adult, animals go through stages resembling or representing successive stages of their remote ancestors. This is a largely discredited evolutionary theory, but one that seems to survive in the mathematics curriculum students are expected to absorb. It is not necessary that every student needing some mathematical understanding be subjected to the arduous training course that leads to an advanced degree in the subject. The only way to break the bonds the traditional math curriculum imposed on students is to adopt the computational power of a CAS as the primary working tool. Indeed, at least one software company is taking a leading role in advocating new "computer-based" curricula at all levels of mathematics education.

Readily available and powerful software can be the basis for a new apprenticeship in STEM (science, technology, engineering, and mathematics) courses. An example why this author found this to be so arose from his early experience with Maple in the classroom at RHIT. Fourier series (sums of sinusoids that approximate a function) were part of the required curriculum. The coefficients in the sums are the values of certain definite integrals. Prior to the availability of Maple, it was observed that students would write a summation symbol, execute some integrals (usually wrong) and submit what was a meaningless assemblage of symbols. In classes after the introduction of Maple whereby students could graph both the function and their proposed Fourier approximations, their behavior changed. They would come with graphs that showed their approximations did not represent the function, now aware that they had made an error. Their question now was: Based on the graphs, could I help them determine just what error had been made. Obviously, a significant conceptual understanding of a Fourier series had been achieved.

In **Figure 1**, the definition of the derivative as the limiting value of the difference quotient is applied to a polynomial. Notice the pedagogical "resequencing" whereby Maple first delivers the end result, then is used to implement the algebraic steps of the derivation. According to **Figure 1**, the polynomial is entered as the function and the definition of the derivative, namely, the limit of the difference quotient, is implemented via Maple's natural mathematical notation, just as it would appear in a textbook. The evaluation is immediate through the Context Menu (a pop-up menu from which options can be selected), or equivalently, via the keyboard. The derivative is also obtained using the notations of both Newton and Leibniz, just to verify

• Context Menu: Assign Function	$f(x) = 2x^3 - 7x^2 + 5x - 4 \xrightarrow{\text{assign as function}} f$
• Calculus palette: Limit operator	$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 6x^2 - 14x + 5$
• Verify that the prime notation returns the derivative.	$f'(x) = 6x^2 - 14x + 5$
• Verify that the differential operator in the Calculus palette also returns the derivative.	$\frac{d}{dx} f(x) = 6x^2 - 14x + 5$
• Write the numerator of the difference quotient. • Context Menu: Expand • Context Menu: Factor • Divide by h, thereby forming the simplified difference quotient. • Context Menu: Evaluate at a Point>$h = 0$	$f(x+h) - f(x)$ $2(x+h)^3 - 7(x+h)^2 + 5h - 2x^3 + 7x^2$ (1) $\overset{\text{expand}}{=}$ $2h^3 + 6h^2x + 6hx^2 - 7h^2 - 14hx + 5h$ (2) $\overset{\text{factor}}{=}$ $h(2h^2 + 6hx + 6x^2 - 7h - 14x + 5)$ (3) $\frac{(3)}{h}$ $2h^2 + 6hx + 6x^2 - 7h - 14x + 5$ (4) $\xrightarrow{\text{evaluate at point}}$ $6x^2 - 14x + 5$ (5)

Figure 1. Applying the definition of the derivative to a polynomial

that these notations mean exactly what the definition claims they mean. Finally, the algebraic steps of the calculation are implemented in Maple, primarily via the Context Menu system.

Another example from the experiences of this author has to do with teaching the definite integral as the limit of an appropriate Riemann sum. Now a Riemann sum is the sum of products of function values and small increments, the sum representing the approximate area under the graph of the function. Before the advent of the hand-held calculator, attempts to teach this concept by doing arithmetic by hand, writing columns of numbers on the blackboard, etc., were totally useless. This approach did no hold students' attention for more than mere minutes, and the concept never got off the ground. With the advent of the hand-held calculator, it seemed that students could be led through all the arithmetic to arrive at a meaningful approximation. Unfortunately, it is impossible to coerce thirty students to uniformly and accurately press all the right keys of a calculator, even if every student has the same calculator. And with no record of what keys were pressed, even these experiments were an abysmal failure.

However, the tools in Maple make this task transparent. To begin, consider the image of the Riemann Sum tutor in **Figure 2**. The graph displayed shows how the area under the graph of a function is approximated by the sum of areas of equi-spaced rectangles under the curve, and very quickly allows the user to see the effect of different partitions, and different strategies for forming the sum.

In **Figure 2**, the Riemann Sums tutor has been applied to the function on the interval . The default partition is ten equal subintervals, and the default sum is a midpoint sum. The actual and approximate areas under the curve are given beneath the graph. As different choices of parameters are made, the graph and the displayed values beneath it are updated. An interface like this allows students to explore the concept of the Riemann sum without the burden of having to do or implement the calculations, let alone draw the corresponding figures.

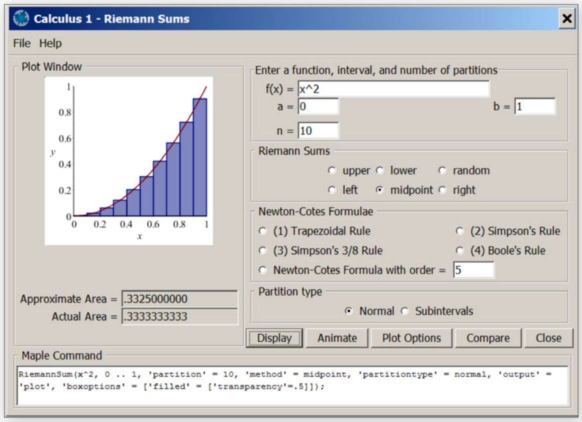

Figure 2. The Riemann Sums tutor applied to $f(x)=x^2$ on the interval [0,1]

The graph in **Figure 2** is drawn by the RiemannSum command that appears at the bottom of the tutor in **Figure 2**. This command will also return the actual Riemann sum, even one with an indeterminate number of subintervals. An example of this is shown in **Figure 3** where, in addition, the value command has been applied to obtain a closed-form of the sum, a form from which the limit as becomes infinite can be determined. This encompasses the definition of the definite integral, and allows the student to see the definition "in action."

$$RS := RiemannSum\left(x^3, x = 0..1, partition = n\right)$$

$$\frac{\sum_{i=0}^{n-1} \frac{\left(i + \frac{1}{2}\right)^3}{n^3}}{n}$$

$$value(RS)$$

$$\frac{-\frac{1}{8n} + \frac{1}{4}n}{n}$$

Figure 3. The **RiemannSum** command with an indeterminate number of subdivisions

Figure 4 again illustrates the definition of the definite integral, this time implemented with Maple's "point-and-click" syntax-free technology. The integrand is defined as a function via the Context Sensitive (pop-up) Menu. The limit and summation operators are implemented as palette templates. The evaluation of the limit of the Riemann left-sum is via the Context Menu. The mathematics in **Figure 3** is exactly the mathematics a textbook would use to express the same concept. That Maple can use the same notation, and that this notation is connected to the underlying computing engine, is the significant observation.

$$f(x) = x^3 \xrightarrow{\text{assign as function}} f$$

$$\lim_{n \to \infty} \sum_{k=0}^{n-1} f(k/n)/n = \frac{1}{4}$$

Figure 4. Syntax-free Maple implementation of the definite integral

Of course, the Riemann sum itself, and its closed-form equivalent, can be obtained with this natural notation and without the burden of specialized syntax. Seen in **Figure 5**, these additional steps are implemented via the Context Menu by simply launching the pop-up menu system on the expressions themselves.

The Riemann sum in **Figure 5** is constructed from a palette template, and is evaluated to a closed-form expression by invoking the Context Menu or its keyboard equivalent. This transition requires some algebraic skill. In the traditional classroom, considerable time and energy must be devoted to this step. By the time students have become sufficiently skilled to evaluate the Riemann sum by hand, the connection of the process to the definition of the definite integral has evaporated. The acquisition of byhand manipulative skills interferes with the absorption of the higher-level concept that the manipulations are supposed to serve. If the tool with which these manipulations were itself equally burdensome to master, it would be a wash - one difficult process replaced by another. That a tool like Maple can easily and naturally replace a tedious manipulation is important for continuity in the exposition of a concept.

$$\sum_{k=0}^{n-1} f(k/n)/n = \frac{1}{4} - \frac{1}{2n} + \frac{1}{4n^2} \xrightarrow{\text{limit w.r.t. n}} \lim_{n \to \infty} \left(\frac{1}{4} - \frac{1}{2n} + \frac{1}{4n^2}\right) = \frac{1}{4}$$

Figure 5. The Riemann sum and its limit

Countless additional examples could be given in support of the contention that not only must the technological tools of a CAS be robust and readily available, but they must also be easy to apply. Examples showing how the tools in Maple satisfy the requirements of robustness and ease-of-use can be found at http://www.maplesoft.com/teachingconcepts/. On this page one finds recorded demonstrations in which standard problems from calculus, differential equations, and linear algebra are solved with a point-and-click paradigm. Indeed, more than 150 such nontrivial examples are listed, and are articulated using built-in tools that require the use of not a single command.

New technologies that improve on existing tools get adopted. It has even happened in mathematics as tables of logarithms sliderules gave way to calculators, then computers. But the really radical step of making a CAS the tool of first recourse in teaching, learning, and doing mathematics has not yet happened, in large part because the learning curve is too steep.

Except for Maple, where the learning curve is relatively flat because of its ease-of-use paradigm.

As the examples in **Figures 1-5** and on the Teaching Concepts page on the Maplesoft site show, significant mathematical exploration and problem-solving can be implemented in Maple without first having to make a capital investment in learning how to use the tool.

This simplicity allows the strategy of resequencing concepts and skills to be realized. Because Maple is easy to use, there is no long build-up to the exploration of a new concept. The mathematical ideas inherent in a topic can be investigated, experienced, manipulated, and learned by using Maple as the working tool, without first having to master a set of manipulative skills. The steps of associated algorithms can be implemented in Maple without the need for a prior mastery of manipulative skills. When the student first grasps the concept and sees how the details relate to this "big picture," mastering the relevant by-hand skills then takes place so much more effectively and efficiently.